Scalar one-loop Feynman integrals in arbitrary space-time dimension

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Why one-loop Feynman integrals? And why in $D = 4 + 2n - 2\epsilon$ dimensions?

Based on [1, 2], I began in 1980 to calculate Feynman integrals, see Mann, Riemann, 1983 [3]: “Effective Flavor Changing Weak Neutral Current In The Standard Theory And Z Boson Decay”

Basics

The seminal papers on 1-loop Feynman integrals:
’t Hooft, Veltman, 1978 [1]: “Scalar oneloop integrals”
Passarino, Veltman, 1978 [2]: “One Loop Corrections for e^+e^- Annihilation into $\mu^+\mu^-$ in the Weinberg Model”

Interest in “modern” developments for the calculation of 1-loop integrals from basically two sides

1. For many-particle calculations, there appear inverse Gram determinants from tensor reductions in the answers.

These $1/G(p_i)$ may diverge, because Gram dets can exactly vanish: $G(p_i) \equiv 0$.

One may perform tensor reductions so that no inverse Grams appear, but one has to buy 1-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon$. See [4, 5]

Interest in “modern” developments for the calculation of 1-loop integrals from basically two sides

2. Higher-order loop calculations need h.o. contributions from ϵ-expansions of 1-loops:
$1/(d-4) = -1/(2\epsilon)$ and $\Gamma(\epsilon) = a/\epsilon + c + \epsilon + \cdots$

A Seminal paper was on ϵ-terms of 1-loop functions:
Nierste, Müller, Böhm, 1992 [6]: “Two loop relevant parts of D-dimensional massive scalar one loop integrals”

1-loop integrals in D dimensions

A general solution in D dimensions was derived in another 2 seminal papers:
Tarasov, 2000 [7] and Fleischer, Jegerlehner, Tarasov, 2003 [8]: “A New hypergeometric representation of one loop scalar integrals in d dimensions”

I was wondering if the results of Fleischer/Jegerlehner/Tarasov (2003) are sufficiently general for practical, black-box applications, and saw a need of creating a software solution in terms of contemporary mathematics.

So we decided to study the issue from scratch in 2 steps:

1st step: Re-derive analytical expressions for scalar one-loop integrals as meromorphic functions of **arbitrary** space-time dimension D

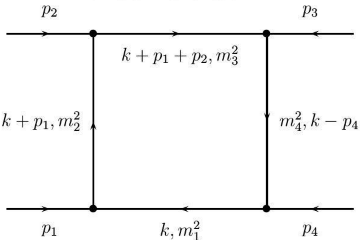
- 2-point functions: Gauss hypergeometric functions ${}_2F_1$ [9]
- 3-point functions: additional Kamp'e de F'eriet functions F_1; there are the Appell functions $F_1, \ldots F_4$ [10]
- 4-point functions: additional Lauricella-Saran functions F_S [11]

2nd step:

Derive the Laurent expansions around the singular points of these functions.

This talk:

- Analytical expressions for self-energies, vertices, boxes
- Numerical checks



$$J_N \equiv J_N(d;\{p_ip_j\},\{m_i^2\}) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}} \tag{1}$$

with

$$D_i = \frac{1}{(k+q_i)^2 - m_i^2 + i\epsilon}. \tag{2}$$

$$\nu_i = 1, \quad \sum_{i=1}^{n} p_i = 0 \tag{3}$$

$$J_n = (-1)^n \Gamma(n-d/2) \int_0^1 \prod_{j=1}^{n} dx_j \delta\left(1 - \sum_{i=1}^{n} x_i\right) \frac{1}{F_n(x)^{n-d/2}} \tag{4}$$

Here, the F-function is the second Symanzik polynomial.

It is derived from the propagators (2),

$$M^2 = x_1 D_1 + \cdots + x_N D_N = k^2 - 2Qk + J. \tag{5}$$

Using $\delta(1 - \sum x_i)$ under the integral in order to transform linear terms in x into quadratic ones, we may obtain:

$$F_n(x) = -(\sum_i x_i) J + Q^2 = \frac{1}{2} \sum_{i,j} x_i Y_{ij} x_j - i\epsilon, \tag{6}$$

The Y_{ij} are elements of the Cayley matrix, introduced for a systematic study of one-loop n-point Feynman integrals e.g. in [12]

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \tag{7}$$

There are $N_n = \frac{1}{2}n(n+1)$ different Y_{ji} for n-point functions: $N_3 = 6, N_4 = 10, N_5 = 15$.

The operator $\mathbf{k}^-$ ...

...will reduce an n-point Feynman integral J_n to an $(n-1)$-point integral J_{n-1} by shrinking the propagator $1/D_k$

$$\mathbf{k}^- \; J_n \quad = \quad \mathbf{k}^- \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j=1}^{n} D_j} \quad = \quad \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j\neq k, j=1}^{n} D_j}. \tag{8}$$

Mellin-Barnes representation

$$\frac{1}{(1+z)^\lambda} \quad = \quad \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(\lambda+s)}{\Gamma(\lambda)} \, z^s \quad = \quad {}_2F_1\left[\begin{matrix} \lambda, b\,; \\ b\,; \end{matrix} \; -z\right]. \tag{9}$$

It is valid if $|\mathrm{Arg}(z)| < \pi$ and the integration contour has to be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda+s)$ are well-separated. The right hand side of (9) is identified as Gauss' hypergeometric function. For more details see [13]).

F-function and Gram and Cayley determinants

Gram and Cayley det's are introduced by Melrose [12] (1965). The Cayley determinant $\lambda_{12\ldots N}$ is composed of the
$Y_{ij} = m_i^2 + m_j^2 - (q_i - q_j)^2$ introduced in (7), and its determinant is:

$$\lambda_n \equiv \lambda_{12\ldots n} = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}. \quad (10)$$

We also define the $(n-1) \times (n-1)$ dimensional Gram determinant $g_n \equiv g_{12\cdots n}$,

$$G_n \equiv G_{12\cdots n} = -\begin{vmatrix} (q_1-q_n)^2 & (q_1-q_n)(q_2-q_n) & \ldots & (q_1-q_n)(q_{n-1}-q_n) \\ (q_1-q_n)(q_2-q_n) & (q_2-q_n)^2 & \ldots & (q_2-q_n)(q_{n-1}-q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1-q_n)(q_{n-1}-q_n) & (q_2-q_n)(q_{n-1}-q_n) & \ldots & (q_{n-1}-q_n)^2 \end{vmatrix}. \quad (1$$

Both determinants are independent of a common shifting of the momenta q_i.
Further, the Gram det G_n is independent of the propagator masses.

Co-factors of the Cayley matrix

One further notation will be introduced, namely that of co-factors of the Cayley matrix.
Also called signed minors in e.g. [12, 14]):

$$\begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n . \quad (12)$$

The signed minors are determinants, labeled by those rows $j_1, j_2, \cdots j_m$ and columns $k_1, k_2, \cdots k_m$ which have been discarded from the definition of the Cayley determinant $()_n$, with a sign convention.

$$\text{sign} \begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n = (-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \times \texttt{Signature}[j_1, j_2, \cdots j_m] \times \texttt{Signature}[k_1, k_2, \cdots k_m] \quad (13)$$

Here, `Signature` (defined like the Mathematica command) gives the sign of permutations needed to place the indices in increasing order.

Cayley matrix, by definition:

$$\lambda_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n . \quad (14)$$

Further, it is (see [15]):

$$-\frac{1}{2}\,\partial_i \lambda_n \equiv -\frac{1}{2}\frac{\partial \lambda_n}{\partial m_i^2} = \begin{pmatrix} 0 \\ i \end{pmatrix}_n . \quad (15)$$

Rewriting the F-function further, exploring the $x_n = 1 - \sum x_i$ ...

The elimination of one of the x_i creates linear terms in $F(x)$.

$$F_n(x) = x^T G_n x + 2H_n^T x + K_n. \tag{16}$$

The $F_n(x)$ may be cast by shifts $x \to (x-y)$ into the form

$$F_n(x) \;=\; (x-y)^T G_n (x-y) + r_n - i\varepsilon \;=\; \Lambda_n(x) + r_n - i\varepsilon \;=\; \Lambda_n(x) + R_n, \tag{17}$$

$$\Lambda_n(x) \;=\; (x-y)^T G_n (x-y), \tag{18}$$

and

$$r_n \;=\; K_n - H_n^T G_n^{-1} H_n \;=\; -\frac{\lambda_n}{g_n} \;=!\; -\frac{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n}{()_n}. \tag{19}$$

The inhomogeneity $R_n = r_n - i\varepsilon$ carries the $i\varepsilon$-prescription.

The linear shifts y_i

The $(n-1)$ components y_i of the vector y appearing here in $F_n(x)$ are:

$$y_i \quad = \quad -\left(G_n^{-1} K_n\right)_i, \quad i \neq n \tag{20}$$

The following relations are also valid:

$$y_i \quad = \quad \frac{\partial r_n}{\partial m_i^2} = -\frac{1}{g_n} \frac{\partial \lambda_n}{\partial m_i^2} = -\frac{\partial_i \lambda_n}{g_n} = \frac{2}{g_n} \left(\begin{array}{c} 0 \\ i \end{array} \right)_n, \quad i = 1 \cdots n. \tag{21}$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled.

We see that the notations for the F-function are finally independent of the choice of the variable which was eliminated by use of the δ-function in the integrand of (4).
The inhomogeneity R_n is the only variable carrying the causal $i\epsilon$-prescription, while e.g. $\Lambda(x)$ and the y_i are by definition real quantities.

The recursion relation for J_n

One may use the Mellin-Barnes relation (9) in order to decompose the integrand of J_n given in (4) as follows:

$$\frac{1}{[F(x)]^{n-\frac{d}{2}}} \equiv \frac{1}{[\Lambda_n(x)+R_n]^{n-\frac{d}{2}}} \equiv \frac{R_n^{-(n-\frac{d}{2})}}{[1+\frac{\Lambda_n(x)}{R_n}]^{n-\frac{d}{2}}}$$
$$= \frac{R_n^{-(n-\frac{d}{2})}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(n-\frac{d}{2}+s)}{\Gamma(n-\frac{d}{2})} \left[\frac{\Lambda_n(x)}{R_n}\right]^s, \qquad (22)$$

for $|\mathrm{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. Further, the integration path in the complex s-plane separates the poles of $\Gamma(-s)$ and $\Gamma(n-\frac{d}{2}+s)$. As a result of (22), the Feynman parameter integral of J_n becomes homogeneous:

$$K_n = \prod_{j=1}^{n-1} \int_0^{1-\sum_{i=j+1}^{n-1} x_i} dx_j \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \int dS_{n-1} \left[\frac{\Lambda_n(x)}{R_n}\right]^s. \qquad (23)$$

The recursion relation for J_n

In order to solve the integral in (23), we consider the differential operator $\hat{P}_n$ [16, 17],

$$\hat{P}_n \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2}(x_i - y_i)\frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n}\right]^s = s \left[\frac{\Lambda_n(x)}{R_n}\right]^s. \tag{24}$$

This eigenvalue relation allows to introduce the operator $\hat{P}_n$ into the integrand of (23):

$$K_n = \int dS_{n-1} \frac{\hat{P}_n}{s} \left[\frac{\Lambda_n(x)}{R_n}\right]^s = \frac{1}{2s}\sum_{i=1}^{n-1}\prod_{k=1}^{n-1}\int_0^{u_k} dx'_k \, (x_i - y_i)\frac{\partial}{\partial x_i}\left[\frac{\Lambda_n(x)}{R_n}\right]^s. \tag{25}$$

After a series of manipulations in order to perform one of the x-integrations – by partial integration, eating the corresponding differential – one arrives at:

$$J_n = \frac{(-1)^n}{2\pi i}\int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(n-\frac{d}{2}+s)\Gamma(s+1)}{2\,\Gamma(s+2)} \left(\frac{1}{R_n}\right)^{n-\frac{d}{2}}$$
$$\times \sum_{i=1}^{n}\left\{\left(\frac{\partial r_n}{\partial m_i^2}\right)\int dS_{n-2}^{(i)}\left[\frac{F_{n-1}^{(i)}}{R_n} - 1\right]^s \right. \tag{26}$$

We stress again that only the R_n carries an $i\epsilon$. Now it is important to eliminate the term (-1) from the combination $(F^{(i)}_{n-1}/R_n - 1)^s$ under the Mellin-Barnes integral over s, because then we arrive at a sum over the n different $(n-1)$-point functions arising from skipping a propagator from the original integral. In fact, this may be arranged using the following relation for $(-z) = F/R - 1$ [18]:

$$\int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(a+s)\,\Gamma(b+s)}{\Gamma(c+s)} (-z)^s \tag{27}$$

$$= \int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(a+b-c-s)\Gamma(c-a+s)\Gamma(c-b+s)}{\Gamma(c-a)\Gamma(c-b)} \, (1-z)^{c-a-b+s},$$

provided that $|\mathrm{Arg}(-z)| < 2\pi$.
We arrive at the following recursion relation:

The recursion relation for 1-loop n-point functions

$$J_n(d,\{q_i,m_i^2\}) = \frac{-1}{2\pi i}\int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s)\,\Gamma(\frac{d-n+1}{2}+s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} R_n^{-s} \times \sum_{k=1}^{n}\left(\frac{1}{r_n}\frac{\partial r_n}{\partial m_k^2}\right) \mathbf{k}^- J_n(d+2s;\{q_i,m_i^2\}). \quad (28)$$

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ prevent the use of the Mellin-Barnes transformation. They are simpler than what we have to do here. Details are given elsewhere.

1-point function, or tadpole

$$J_1(d;m^2) = \int \frac{d^dk}{i\pi^{d/2}}\frac{1}{k^2-m^2+i\epsilon} = -\frac{\Gamma(1-d/2)}{(m^2-i\epsilon)^{1-d/2}}. \quad (29)$$

Comments

1. In Tarasov 2003 [8], a recursion was derived where our Mellin-Barnes integral is replaced by an infinite sum to be solved. Formulae for 2,3,4-point functions are given.

2. A 4-point function is a 3-fold integral. With AMBRE, we get up to 15-fold integrals instead.

3. See Johann Usovitsch's talk: Integrand is equally integrable for Euklidean and Minkoswkian cases.
No Gram=0 problem.

The 2-point function

From our recursion relation (28), taken at $n = 2$ and using the expression (29) with $d \to d + 2s$ for the one-point functions under the integral, one gets the following representation:

$$J_2(D; q_1, m_1^2, q_2, m_2^2) = \frac{e^{\epsilon\gamma_E}}{2\pi i} \int\limits_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma\left(\frac{D-1}{2}+s\right)\,\Gamma(s+1)}{2\,\Gamma\left(\frac{D-1}{2}\right)} R_2^s \times \left[\frac{1}{r_2} \frac{\partial r_2}{\partial m_2^2} \frac{\Gamma\left(1-\frac{D+2s}{2}\right)}{(m_1^2)^{1-\frac{D+2s}{2}}} + (m_1^2 \leftrightarrow m_2^2) \right]. \quad (30)$$

One may close the integration contour of the MB-integral in (30) to the right, apply the Cauchy theorem and collect the residua originating from two series of zeros of arguments of Γ-functions at $s = m$ and $s = m - d/2 - 1$ for $m \in \mathbb{N}$.

The first series stems from the MB-integration kernel, the other one from the dimensionally shifted 1-point functions.

And then summing up in terms of Gauss' hypergeometric functions.

The 2-point function (slightly rewritten), $R_2 \equiv R_{12}$

$$J_2(d; Q^2, m_1^2, q_2, m_2^2) = -\frac{\Gamma\left(2-\frac{d}{2}\right)}{(d-2)} \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{\partial_2 R_2}{R_2} \left[(m_1^2)^{\frac{d}{2}-1} {}_2F_1\left[\begin{matrix} 1, \frac{d}{2}-\frac{1}{2}; \\ \frac{d}{2}; \end{matrix} \frac{m_1^2}{R_2} \right] + \frac{R_2^{\frac{d}{2}-1}}{\sqrt{1-\frac{m_1^2}{R_2}}} \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)} \right] + (m_1^2 \leftrightarrow m_2^2) \qquad (31)$$

The representation (31) is valid for $\left|\frac{m_1^2}{r_{12}}\right| < 1$, $\left|\frac{m_2^2}{r_{12}}\right| < 1$ and $\mathcal{R}\mathrm{e}(\frac{d-2}{2}) > 0$.

The result is in agreement with Eqn. (53) of of Tarasov et al. (2003) [8].

The 3-point function

According to the master formula (28), we can write the massive 3-point function as a sum of three terms:

$$J_3 = J_{123} + J_{231} + J_{312}, \tag{32}$$

using the representation for e.g. J_{123}

$$J_{123}(d, \{q_i, m_i^2\}) = -\frac{e^{\epsilon\gamma_E}}{2\pi i}\int_{-i\infty}^{+i\infty} ds \, \frac{\Gamma(-s)\,\Gamma(\frac{d-2+2s}{2})\Gamma(s+1)}{2\,\Gamma(\frac{d-2}{2})} R_3^{-s} \times \frac{1}{r_3}\frac{\partial r_3}{\partial m_3^2} \, J_2(d+2s; q_1, m_1^2, q_2, m_2^2). \tag{33}$$

Here, $J_2(d+2s; q_1, m_1^2, q_2, m_2^2)$ is given by (31), taken at $d+2s$ dimensions. By performing the Mellin-Barnes integrals, one gets three terms, each consisting of eight series, from taking the residues by closing the integration contours to the right; one of the three terms is:

$$
\begin{aligned}
J_{123} &= \Gamma\left(2-\frac{d}{2}\right) R_{123}^{\frac{d}{2}-2} \times b_{123} \\
&- \frac{\sqrt{\pi}\,\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)} \frac{\partial_3\lambda_{123}}{\lambda_{123}} \frac{R_{12}^{\frac{d}{2}-1}}{4\lambda_{12}} \left[\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} + \frac{\partial_1\lambda_{12}}{\sqrt{1-\frac{m_2^2}{R_{12}}}}\right] \times {}_2F_1\left[\begin{matrix}\frac{d-2}{2}, 1; \\ \frac{d-1}{2};\end{matrix} \frac{R_{12}}{R_{123}}\right] \\
&+ \frac{2}{d-2}\Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3\lambda_{123}}{\lambda_{123}} \times \left[\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} \frac{(m_1^2)^{\frac{d}{2}-1}}{4\lambda_{12}} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_{123}}, \frac{m_1^2}{R_{12}}\right)\right. \\
&\left. + (m_1^2 \leftrightarrow m_2^2)\right],
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
b_{123} &= -\frac{1}{2g_{12}} \frac{\partial_3\lambda_{123}}{\lambda_{123}} \left(\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} + \frac{\partial_1\lambda_{12}}{\sqrt{1-\frac{m_2^2}{R_{12}}}}\right) {}_2F_1\left[\begin{matrix}1, 1; \\ \frac{3}{2};\end{matrix} \frac{R_{12}}{R_{123}}\right] \\
&- \frac{\partial_3\lambda_{123}}{\lambda_{123}} \left\{\frac{\partial_2\lambda_{12}}{\sqrt{1-\frac{m_1^2}{R_{12}}}} \frac{m_1^2}{4\lambda_{12}} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_{123}}, \frac{m_1^2}{R_{12}}\right) + (m_1^2 \leftrightarrow m_2^2)\right\},
\end{aligned}
\tag{35}
$$

where $\partial_i \lambda_j \ldots$ is defined in (21). The representation (32) is valid for Re$(d - 2/2) > 0$. The conditions $|m_i^2/R_{ij}| < 1$, $|R_{ij}/R_{ijk}| < 1$ had to be met during the derivation. The result may be analytically continued in a straightforward way, however, in the complete complex domain.

The functions ${}_2F_1$ and F_1 of the b_{ijk}-terms are met by setting $d = 4$ in the corresponding functions J_{ijk} of the general J_3.

An alternative writing of $J_3 = J_{123} + J_{231} + J_{312}$ is, with $R_3 = R_{123}, R_2 = R_{12}$ etc. here:

The massive vertex

$$
\begin{aligned}
J_{123} &= \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_2}{2\sqrt{1-m_1^2/r_2}} \\
&\left[- R_2^{d/2-2} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)} {}_2F_1\left[\begin{matrix} \frac{d-2}{2}, 1\,; \\ \frac{d-1}{2}\,; \end{matrix} \; \frac{R_2}{R_3} \right] + R_3^{d/2-2} \, {}_2F_1\left[\begin{matrix} 1, 1\,; \\ 3/2\,; \end{matrix} \; \frac{R_2}{R_3} \right] \right] \\
&+ \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{m_1^2}{4\sqrt{1-m_1^2/r_2}} \\
&\left[+ \frac{2(m_1^2)^{d/2-2}}{d-2} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2} \right) - R_3^{d/2-2} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) \right] \\
&+ \; (m_1^2 \leftrightarrow m_2^2)
\end{aligned}
$$

For $d \to 4$, both the $[...]$ approach zero.

So the J_3 is finite in this limit, as it should be for massive 3-point function.

For the 3-point function, we look at the expression $J_{123} + J_{231} + J_{312}$.

We should agree with Eqn. (74) to (76) of Tarasov (2003).

Our terms with d-dimensional F_1 and ${}_2F_1$ do agree exactly, but $b_{123} + b_{231} + b_{312}$ looks quite different.

Tarasov (2003) [8], Eqns. (73) and (75)

There are kinematic conditions on internal momenta $q_{ij}^2 = (q_i - q_j)^2$ to be respected; the b_3-term of Tarasov becomes:

$$J_3(b_3) = \theta(-G_3) \times \theta(q_{ij}^2) \times \theta(\frac{m_i^2}{r_3} - 1) \times \frac{\Gamma(2-d/2)}{\lambda_3} \left(2^{3/2}\, \pi\, \sqrt{-G_3}\;\, R_3{}^{d/2-1}\right) \quad (36)$$

Otherwise:

$$J_3(b_3) = b_3 = 0. \quad (37)$$

Numerics for 3-point functions, table 1

$[p_i^2]$, $[m_i^2]$	[+100, +200, +300], [10, 20, 30]	
G_{123} λ_{123} m_i^2/r_{123} m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	–160000 –8860000 –0.180587, –0.361174, –0.541761 –0.97561, –1.95122, –2.92683 –0.39801, –0.79602, –1.19403 –0.180723, –0.361446, –0.542169 (0.019223879 – 0.007987267 I)	
$\sum b_3$-terms J_3(TR)	0 (0.019223879 – 0.007987267 I)	
b_3-term $b_3 + \sum J$-terms	(–0.089171509 + 0.069788641 I) (–0.012307377 – 0.009301346 I)	+ (0.022214414)/eps
J_3(OT)	$\sum J$-terms, b_3-term $\to$ 0, OK	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	0.0192238790286244077-0.00798726725497102795 i	

Table 1: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities (external momenta shown here!) suggest that, according to Eqn. (73) in Tarasov (2003) [8], one has to set $b_3 = 0$.
Although b_3 of [8] deviates from our vanishing value, it has to be set to zero, $b_3 \to 0$.
The results of both calculations for J_3 agree for this case.

Numerics for 3-point functions, table 2

$[p_i^2]$, $[m_i^2]$	[-100, +200, -300], [10, 20, 30]	
G_{123} λ_3 m_i^2/r_3 m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	480000 -19300000 0.248705, 0.497409, 0.746114 0.248447, 0.496894, 0.745342 -0.39801, -0.79602, -1.19403 0.104895, 0.20979, 0.314685 (-0.012307377 - 0.056679689 I)	+ (+ 0.012825498 I)/eps
$\sum b_3$-terms J_3(TR)	(+ 0.047378343 I) (-0.012307377 - 0.009301346 I)	- (+ 0.012825498 I)/eps
b_3-term $b_3+\sum J$-terms	(+ 0.047378343 I) (-0.012307377 - 0.009301346 I)	- (+ 0.012825498 I)/eps
J_3(OT)	$\sum J$-terms, b_3-term$\to$0, gets wrong	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	-0.0123073773677820630 - 0.0093013461700863289 i	

Table 2: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [8], one has to set $b_3 = 0$. Further, we have set in the numerics for eq. (75) of Tarasov2003 [8] that Sqrt[-g123 + I*epsil], what looks counter-intuitive for a "momentum"-like function.

Both results agree if we *do not* set Tarasov's $b_3 \to 0$.

Numerics for 3-point functions, table 3

p_i^2 m_i^2	–100,–200,–300 10,20,30	
G_{123} λ_{123} m_i^2/r_{123} m_i^2/r_{12} m_i^2/r_{23} m_i^2/r_{31} $\sum J$-terms	–160000 15260000 0.104849, 0.209699, 0.314548 0.248447, 0.496894, 0.745342 0.133111, 0.266223, 0.399334 0.104895, 0.20979, 0.314685 (0.0933877 – 0 I)	– (0.0222144 – 0 I)/eps
$\sum b$-terms J_3(TR)	-0.101249 (–0.00786155 – 0 I)	+ 0.0222144/eps
b_3 b_3+J-terms	(-0.101249 + 0 I) (–0.007861546 + 0 I)	+ (0.0222144 + 0 I)/eps
J_3(OT)	b_3+J-terms $\to$ OK	
MB suite	–0.007862014, 5.002549159*10-6, 0	
(-1)*fiesta3	–(0.007862)	+ (6*10-6 + 6*10-6 I pm10)
LoopTools/FF, ϵ^0	–0.00786154613229082290	

Table 3: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$.

Agreement with Tarasov (2003).

Numerics for 3-point functions, table 4

p_i^2	+100, –200, +300	
m_i^2	10, 20, 30	
G_{123}	480000	
λ_{123}	4900000	
m_i^2/r_{123}	–0.979592, –1.95918, –2.93878	
m_i^2/r_{12}	–0.97561, –1.95122, –2.92683	
m_i^2/r_{23}	0.133111, 0.266223, 0.399334	
m_i^2/r_{31}	–0.180723, –0.361446, –0.542169	
$\sum J$-terms	(0.006243624 - 0.018272524 I)	
$\sum b_3$-terms	0	
J_3(TR)	(0.006243624 - 0.018272524 I)	
b_3-term	(0.040292491 + 0.029796253 I)	+ (- 0.012825498 I)/eps
$b_3+\sum J$-terms	(-0.012307377 - 0.009301346 I)	+ (4*-18 - 6*-18 I)/eps
J_3(OT)	$\sum J$-terms, b_3-term→0, OK	
MB suite		
(-1)*fiesta3	-(-0.006322 + 0.014701 I)	+ (0.000012 + 0.000014 I) pm
LoopTools/FF, ϵ^0	0.00624362477277410 - 0.01827252404872805 i	

Table 4: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [8], one has to set $b_3 = 0$.
Agreement with Tarasov (2003) due to setting $b_3 = 0$ there.

The 4-point function

According to the master formula (28), we can write the massive 4-point function as a sum of four terms:

$$J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}, \quad (38)$$

Each of the four terms has the structure

$$J_{1234} = \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-3}{2}\right)} \times (r_{1234})^{\frac{d}{2}-2} \times \hat{b}_{1234} + \Gamma\left(2-d/2\right) \times \hat{J}^{d}_{1234} \quad (39)$$

The pre-factor is singular: $\Gamma(2-d/2) = 1/\epsilon + \cdots$ for $d \geq 4-2\epsilon$.

We agree for $\hat{J}^{d}_{1234}$ etc. with Tarasov (2003) [?].
For the b_4-term, the situation is a bit unclear.

The boundary term $\hat{b}_{1234}$ is independent of d:

$$\begin{aligned}
\hat{b}_{1234} \quad = \quad & \frac{1}{2}\left(\frac{b_{123}}{r_{1234}}\frac{\partial r_{1234}}{\partial m_4^2}\right)\frac{\sqrt{\pi}}{\sqrt{1-r_{123}/r_{1234}}} \\
& + \sqrt{\pi}\left(\frac{1}{r_{1234}}\frac{\partial r_{1234}}{\partial m_4^2}\right)\left(\frac{1}{r_{123}}\frac{\partial r_{123}}{\partial m_3^2}\right)\left(\frac{1}{4g_{12}}\right)\times \\
& \times\left[\frac{\partial_2\lambda_{12}}{\sqrt{1-m_1^2/r_{12}}}+\frac{\partial_1\lambda_{12}}{\sqrt{1-m_2^2/r_{12}}}\right]\left(\frac{1}{\sqrt{1-r_{12}/r_{123}}}\right)F_1\left(\frac{1}{2};1,\frac{1}{2};\frac{3}{2};\frac{r_{12}}{r_{1234}},\frac{r_{12}}{r_{123}}\right) \qquad (40) \\
& + \sqrt{\pi}\left(\frac{1}{r_{1234}}\frac{\partial r_{1234}}{\partial m_4^2}\right)\left(\frac{1}{r_{123}}\frac{\partial r_{123}}{\partial m_3^2}\right)\times \\
& \times\left[\left(\frac{\partial_2\lambda_{12}}{1-\frac{m_1^2}{r_{12}}}\right)\left(\frac{m_1^2}{8\lambda_{12}}\right)\left(\frac{r_{123}}{r_{123}-m_1^2}\right)\right. \\
& \left.\times F_S\left(\frac{1}{2},1,1;1,1,\frac{1}{2};2,2,2;\frac{m_1^2}{r_{1234}},\frac{m_1^2}{m_1^2-r_{123}},\frac{m_1^2}{m_1^2-r_{12}}\right)+(1\leftrightarrow 2)\right] \\
& +(2,3,1)+(3,1,2).
\end{aligned}$$

The boundary term b_4 has not been exactly defined in [?], concerning the kinematica conditions. We did not perform massive numerical tests.

and

$$
\begin{aligned}
\hat{J}_{1234} = \; & (r_{1234})^{\frac{d}{2}-2} \times b_{1234} \\
& - \frac{1}{2} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) {}_2F_1 \left[\begin{matrix} \frac{d-3}{2}, 1 ; \\ \frac{d}{2} - 1 ; \end{matrix} \; \frac{r_{123}}{r_{1234}} \right] \times (r_{123})^{\frac{d}{2}-2} \times b_{123} \\
& + \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(\frac{d-1}{2})} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left(\frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \left[\frac{\partial_2 \lambda_{12}}{\sqrt{1-\frac{m_1^2}{r_{12}}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1-\frac{m_2^2}{r_{12}}}} \right] \times \\
& \times \left(\frac{r_{12}^{\frac{d}{2}-1}}{8\lambda_{12}} \right) \left(\frac{1}{\sqrt{1-\frac{r_{12}}{r_{123}}}} \right) F_1 \left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{r_{12}}{r_{1234}}, \frac{r_{12}}{r_{123}} \right) \\
& - \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(\frac{d}{2})} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left(\frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \times \\
& \times \Bigg[\left(\frac{r_{123}}{r_{123}-m_1^2} \right) \left(\frac{r_{12}}{r_{12}-m_1^2} \right) \left(\frac{\partial_2 \lambda_{12}}{8\lambda_{12}} \right) (m_1^2)^{\frac{d}{2}-1} \times \\
& \times F_S \left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \frac{m_1^2}{r_{1234}}, \frac{m_1^2}{m_1^2 - r_{123}}, \frac{m_1^2}{m_1^2 - r_{12}} \right) + (1 \leftrightarrow 2) \Bigg] \\
& + (2,3,1) + (3,1,2).
\end{aligned}
\tag{41}
$$

An alternative writing of $J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}$ is, with $R_4 = R_{1234}, R_3 = R_{123}, R_2 = R_{12}$ etc. here:

The massive box function

$$
\begin{aligned}
J_{1234} \quad = \quad & \Gamma\left(2-\frac{d}{2}\right)\frac{\partial_4 r_4}{r_4}\Bigg\{ \\
& \left[\frac{b_{123}}{2}\left(-R_3^{d/2-2}\,{}_2F_1\left[\begin{matrix}\frac{d-3}{2},1; & \frac{R_2}{R_3}\\ \frac{d}{2}-1; & \end{matrix}\right] + R_4^{d/2-2}\sqrt{\pi}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\,{}_2F_1(d\to 4)\right)\right] \\
& + \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\frac{\sqrt{\pi}}{4}\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{\sqrt{1-m_1^2/R_2}}\,{}_2F_1\left[\begin{matrix}1/2,1; & \frac{R_2}{R_3}\\ 1; & \end{matrix}\right] \\
& \left[+\frac{R_2^{d/2-2}}{d-3}F_1\left(\frac{d-3}{2};1,\frac{1}{2};\frac{d-1}{2};\frac{R_2}{R_4},\frac{R_2}{R_3}\right) - R_4^{d/2-2}F_1\left(d\to 4\right)\right] \\
& \frac{m_1^2}{8}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{r_3}{r_3-m_1^2}\frac{r_2}{r_2-m_1^2} \\
\Bigg[-(m_1^2)^{d/2-2}&\frac{\Gamma\left(\frac{d}{2}-3/2\right)}{\Gamma\left(\frac{d}{2}\right)}F_S(d/2-3/2,1,1,1,1,d/2,d/2,d/2,d/2,\frac{m_1^2}{R_4},\frac{m_1^2}{m_1^2-R_3},\frac{m_1^2}{m_1^2-R_2}) \\
& + R_4^{d/2-2}\sqrt{\pi}F_S(d\to 4)\Bigg] \quad + \quad (m_1^2 \leftrightarrow m_2^2)\Bigg\}
\end{aligned}
\tag{42}
$$

For $d \to 4$,all three $[...]$ approach zero.
So that the massive J_4 gets finite then: OK.

Summary

- **We derived a new recursion relation for one-loop scalar Feynman integrals:** self-energies, vertices, boxes etc.
- The condition $\nu_i = 1$ seems to be essential for that.
- A generalization to multiloops seems to be not straightforward or impossible.
- **Solving the recursions in terms of special functions reproduces essential parts of the results of Tarasov et al. from 2003.**
- **Concerning their b_3-terms, we see a need of improvement compared to their paper, if their result is not just wrong in some kinematical situations.** Our conclusions concerning that depend somewhat on an interpretation of their text.
- **We derived a new series of Mellin-Barnes representations: 1-dimensional for self-energies, 2-dim. for vertices, and 3-dimensional for box diagrams** for the most general kinematics. Compared to dim=3, 5, 9 respectively, in the "conventional" Mellin-Barnes-approach.
  This is worked out by Johann Usovitsch.
  Again, we see no direct generalization to multi-loops.
- The special case of **vanishing Gram determinant** $G_n = 0$ is not covered. But small Gram determinants are, and one has to take measures to get reasonable numerics. $\rightarrow$ Small Gram dets are very interesting, but nothing is done.

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